

# Super- $\tau_3$ QED and the dimensional reduction of $N=1$ super-QED $_{2+2}$

*M. A. De Andrade\** and *O. M. Del Cima<sup>†</sup>*

Centro Brasileiro de Pesquisas Físicas (CBPF)

Departamento de Teoria de Campos e Partículas (DCP)

Rua Dr. Xavier Sigaud, 150 - Urca

22290-180 - Rio de Janeiro - RJ - Brazil.

## Abstract

In this work the supersymmetric gauge invariant action for the massive Abelian  $N=1$  super-QED $_{2+2}$  in the Atiyah-Ward space-time ( $D=2+2$ ) is formulated. The questions concerning the scheme of the gauge invariance in  $D=2+2$  by means of gauging the massive  $N=1$  super-QED $_{2+2}$  are investigated. We study how to ensure the gauge invariance at the expenses of the introduction of a complex vector superfield. We discuss the Wess-Zumino gauge and thereupon we conclude that in this gauge, only the imaginary part of the complex vector field,  $B_\mu$ , gauges a  $U(1)$ -symmetry, whereas its real part gauges a Weyl symmetry. We build up the gauge invariant massive term by introducing a pair of chiral and anti-chiral superfields with opposite  $U(1)$ -charges. We carry out a dimensional reduction *à la* Scherk of the massive  $N=1$  super-QED $_{2+2}$  action from  $D=2+2$  to  $D=1+2$ . Truncations are needed in order to suppress non-physical modes and one ends up with a parity-preserving  $N=1$  super-QED $_{1+2}$  (rather than  $N=2$ ) in  $D=1+2$ . Finally, we show that the  $N=1$  super-QED $_{1+2}$  we have got is the supersymmetric version of the  $\tau_3$ QED .

---

\*Internet e-mail: marco@cbpfsu1.cat.cbpf.br

<sup>†</sup>Internet e-mail: oswaldo@cbpfsu1.cat.cbpf.br

# 1 Introduction

The idea of space-times with several time components and indefinite signature has been taken seriously into account since a self-dual Yang-Mills theory in 4-dimensions [1] has been related to the Atiyah-Ward conjecture [2]. This theory is considered as a potential source for all integrable models in lower dimensions, after some appropriate dimensional reduction (DR) scheme is adopted.

Recently, it has been pointed out by Oogury and Vafa [3] that the consistent backgrounds for  $N=2$  string propagation correspond to self-dual gravity (SDG) configurations in the case of closed  $N=2$  strings, and self-dual Yang-Mills (SDYM) configurations, coupled to gravity, in the case of  $N=2$  heterotic strings in four and lower dimensions. This result has been reconfirmed by Gates and Nishino [4] on the basis of  $\beta$ -function calculations for the Yang-Mills sector of the  $N=2$  heterotic string. More recently, Gates, Ketov and Nishino [5] have noticed the existence of Majorana-Weyl spinors in the Atiyah-Ward space-time, *i.e.*  $D=2+2$ , and an  $N=1$  self-dual supersymmetric Yang-Mills (SDSYM) theory and a self-dual supergravity (SDSG) model were formulated for the first time. Afterwards, an  $N=2$  self-dual supersymmetric Yang-Mills theory and  $N=2$  and  $N=4$  self-dual supergravities have been formulated; in view of these results, it was also conjectured that the  $N=2$  superstrings have no possible counterterms at quantum level to all orders in string loops [6].

The evidence for the close relationship between supersymmetric Chern-Simons (SCS) theory and integrable models or topological theories gives enough motivation to concentrate efforts in trying to understand more about field theories in 3-dimensions. It has already been shown that the  $N=1$  and  $N=2$  SCS theories in  $D=1+2$  are directly generated by the  $N=1$  and  $N=2$  SDSYM theories in  $D=2+2$  by a suitable dimensional reduction and truncation [7].

Since over the past years 3-dimensional field theories [8] have been shown to play a central rôle in connection with the behaviour of 4-dimensional theories at finite temperature [9], as well as in the description of a number of problems in Condensed Matter Physics [10, 11, 12], it seems reasonable to devote some attention to understand some peculiar features of gauge field dynamics in 3 dimensions. Also, the recent result on the Landau gauge finiteness of Chern-Simons theories is a remarkable property that makes 3-dimensional gauge theories so attractive [13]. Very recently, this line of investigation has been well-motivated in view of the possibilities of providing a gauge-theoretical foundation for the description of Condensed Matter phenomena, such as high- $T_c$  superconductivity [11], where the  $\text{QED}_3$  and  $\tau_3\text{QED}_3$  [11, 12] are some of the theoretical approaches that been forwarded as an attempt to understand more deeply about high- $T_c$  materials.

The main purpose of this paper is to build up a superspace action that describes a massive Abelian gauge model in  $D=2+2$ , namely, the  $N=1$  version of  $\text{QED}_{2+2}$ . Our work is organized as follows. In Section 2, we give the details of the formulation of the  $N=1$  supersymmetry in the Atiyah-Ward space-time. The discussion and the explicit construction of an Abelian gauge model with  $N=1$  supersymmetry in  $D=2+2$  is the content of Section 3. Here, we take massive matter fields, but the massless case is also contemplated as a particular case of the former.

We show in Section 4 that, in carrying out a dimensional reduction *à la* Scherk [14, 15] of the massive  $N=1$  super-QED<sub>2+2</sub> action to  $D=1+2$  space-time dimensions, truncations are needed in order to suppress non-physical modes and we end up with a parity-preserving  $N=1$  super- $\tau_3$ QED (the supersymmetric version of the  $\tau_3$ QED) [16], whose spectrum is free from tachyons and ghosts at tree-level. Finally, in Section 5, we draw our general conclusions and present our prospects for future work. Two appendices follow: the relevant aspects of spinors in  $D=2+2$  and our notational conventions to work in  $D=2+2$  are listed in the Appendix A. In the Appendix B, the conventions for  $D=1+2$  and some rules for dimensional reduction are collected. The metric adopted throughout this work for the Atiyah-Ward space-time is  $\eta_{\mu\nu} = (+, -, -, +)$ ,  $\mu, \nu=(0,1,2,3)$ .

## 2 $N=1$ supersymmetry and superfields in Atiyah-Ward space-time

Ordinary space-time can be defined as the coset space (Poincaré group)/(Lorentz group). Similarly, globally flat superspace can be defined as the coset space (super-Poincaré group)/(Lorentz group): its points are the orbits which the Lorentz group sweeps out in the super-Poincaré group. Elements of superspace are labeled by the Atiyah-Ward space-time coordinates;  $x^\mu$ , where  $\mu=(0, 1, 2, 3)$ , and the fermionic coordinates;  $\theta^\alpha$  and  $\tilde{\theta}^{\dot{\alpha}}$ , where  $\alpha=(1, 2)$  and  $\dot{\alpha}=(\dot{1}, \dot{2})$ . The fermionic coordinates  $\theta$  and  $\tilde{\theta}$  are Majorana-Weyl spinors.

Superfields are analytic functions of superspace coordinates, which should be understood in terms of their power series expansions in  $\theta$  and  $\tilde{\theta}$  with coefficients which are themselves local fields over Minkowski space [17, 18].

A compact and very useful technique for working out representations of the supersymmetry algebra on fields was proposed by Salam and Strathdee [19, 20]: superfields in superspace. It is particularly useful for  $N=1$  theories, where their superfield structure is completely known. The well-known algebra fulfilled by the generators of the supersymmetry in  $D=2+2$ ,  $P_\mu$ ,  $Q_\alpha$  and  $\tilde{Q}_{\dot{\alpha}}$ , is given by <sup>1</sup>

$$\begin{aligned} \{Q_\alpha, \tilde{Q}_{\dot{\alpha}}\} &= 2 \sigma_{\alpha\dot{\alpha}}^\mu P_\mu, \quad \{Q_\alpha, Q_\beta\} = \{\tilde{Q}_{\dot{\alpha}}, \tilde{Q}_{\dot{\beta}}\} = 0 \\ \text{and } [Q_\alpha, P_\mu] &= [\tilde{Q}_{\dot{\alpha}}, P_\mu] = 0. \end{aligned} \quad (1)$$

The transformation law for a superfield,  $F(x, \theta, \tilde{\theta})$ , is defined as follows :

$$\delta F \equiv i (\varepsilon Q + \tilde{\varepsilon} \tilde{Q}) F \quad (2)$$

where the parameters  $\varepsilon^\alpha$  and  $\tilde{\varepsilon}^{\dot{\alpha}}$  are Majorana-Weyl spinors as the same for the supercharges,  $Q_\alpha$  and  $\tilde{Q}_{\dot{\alpha}}$ , that are given by

$$Q_\alpha = -i(\partial_\alpha + i\tilde{\partial}_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}) \quad \text{and} \quad \tilde{Q}_{\dot{\alpha}} = -i(\tilde{\partial}_{\dot{\alpha}} + i\tilde{\partial}_{\alpha\dot{\alpha}}\theta^\alpha). \quad (3)$$

---

<sup>1</sup>For notation and conventions in  $D=2+2$  see the Appendix A.

The translations in superspace which result from the supersymmetry transformations of the superfield  $F(x, \theta, \tilde{\theta})$  are presented below

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + i \varepsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} + i \tilde{\varepsilon}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \theta^\alpha, \\ \theta^\alpha &\longrightarrow \theta^\alpha + \varepsilon^\alpha \quad \text{and} \quad \tilde{\theta}^{\dot{\alpha}} \longrightarrow \tilde{\theta}^{\dot{\alpha}} + \tilde{\varepsilon}^{\dot{\alpha}}. \end{aligned} \quad (4)$$

The covariant derivatives,  $D_\alpha$  and  $\tilde{D}_{\dot{\alpha}}$ , are such that the application of them on a superfield, i.e.,  $D_\alpha F$  and  $\tilde{D}_{\dot{\alpha}} F$ , are covariant under supersymmetry transformations, this means that they are also superfields, and these derivatives are such that

$$D_\alpha = \partial_\alpha - i \tilde{\theta}_{\alpha\dot{\alpha}} \tilde{\theta}^{\dot{\alpha}} \quad \text{and} \quad \tilde{D}_{\dot{\alpha}} = \tilde{\partial}_{\dot{\alpha}} - i \tilde{\theta}_{\dot{\alpha}\alpha} \theta^\alpha, \quad (5)$$

where they fulfil the following algebra

$$\begin{aligned} \{D_\alpha, \tilde{D}_{\dot{\alpha}}\} &= -2i \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, \quad \{D_\alpha, D_\beta\} = \{\tilde{D}_{\dot{\alpha}}, \tilde{D}_{\dot{\beta}}\} = 0 \\ \text{and} \quad [D_\alpha, \partial_\mu] &= [\tilde{D}_{\dot{\alpha}}, \partial_\mu] = 0. \end{aligned} \quad (6)$$

A chiral superfield,  $\Psi$ , is characterized by the covariant condition  $\tilde{D}_{\dot{\alpha}} \Psi = 0$ ; therefore, it follows that this superfield may be generally parametrized as

$$\Psi(x, \theta, \tilde{\theta}) = e^{i\tilde{\theta}\tilde{\theta}\theta} \left[ A(x) + i\theta\psi(x) + i\theta^2 F(x) \right], \quad (7)$$

where  $A$  is a complex scalar,  $\psi$  is a Weyl spinor and  $F$  is a complex scalar auxiliary field. The superfield,  $\Psi^\dagger$ , that arises from the Hermitian conjugation of the chiral superfield,  $\Psi$ , is also a chiral superfield (peculiarity of  $D=2+2$ ), contrary to what happens in  $D=1+3$ . The superfield  $\Psi^\dagger$  is given by

$$\Psi^\dagger(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ A^*(x) + i\theta\psi^c(x) + i\theta^2 F^*(x) \right], \quad (8)$$

where we used the relations (A.31) and (A.32) of Appendix A.

An anti-chiral superfield,  $\tilde{X}$ , is such that it satisfies the constraint  $D_\alpha \tilde{X} = 0$ , and may be written as follows

$$\tilde{X}(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ B(x) + i\tilde{\theta}\tilde{\chi}(x) + i\tilde{\theta}^2 G(x) \right], \quad (9)$$

where  $B$  is a complex scalar,  $\tilde{\chi}$  is a Weyl spinor and  $G$  is a complex scalar auxiliary field. Analogously to the previous case (by using (A.31) and (A.32)), the anti-chiral superfield  $\tilde{X}^\dagger$  is

$$\tilde{X}^\dagger(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ B^*(x) + i\tilde{\theta}\tilde{\chi}^c(x) + i\tilde{\theta}^2 G^*(x) \right]. \quad (10)$$

The rigid supersymmetry transformation law defined by eq.(2) yields for the components of  $\Psi$  and  $\tilde{X}$  the following transformations :

$$\left\{ \begin{array}{l} \delta A = i\varepsilon^\alpha \psi_\alpha \\ \delta \psi_\alpha = 2\varepsilon_\alpha F - 2\tilde{\varepsilon}^{\dot{\alpha}} \tilde{\partial}_{\dot{\alpha}\alpha} A \\ \delta F = i\tilde{\varepsilon}^{\dot{\alpha}} \tilde{\partial}_{\dot{\alpha}}{}^\alpha \psi_\alpha \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \delta B = i\tilde{\varepsilon}^{\dot{\alpha}} \tilde{\chi}_{\dot{\alpha}} \\ \delta \tilde{\chi}_{\dot{\alpha}} = 2\tilde{\varepsilon}_{\dot{\alpha}} G - 2\varepsilon^\alpha \partial_{\alpha\dot{\alpha}} B \\ \delta G = i\varepsilon^\alpha \partial_\alpha{}^{\dot{\alpha}} \tilde{\chi}_{\dot{\alpha}} \end{array} \right. . \quad (11)$$

Bearing in mind the necessity of the formulation of a supersymmetric gauge theory in the Atiyah-Ward space-time, we are compelled to introduce a *complex* vector superfield (a vector superfield without the reality constraint),  $V$  :

$$V(x, \theta, \tilde{\theta}) = C(x) + i\theta\zeta(x) + i\tilde{\theta}\tilde{\eta}(x) + \frac{1}{2}i\theta^2 M(x) + \frac{1}{2}i\tilde{\theta}^2 N(x) + \frac{1}{2}i\theta\sigma^\mu\tilde{\theta}B_\mu(x) - \frac{1}{2}\tilde{\theta}^2\theta\lambda(x) - \frac{1}{2}\theta^2\tilde{\theta}\tilde{\rho}(x) - \frac{1}{4}\theta^2\tilde{\theta}^2 D(x) , \quad (12)$$

where  $C$ ,  $M$ ,  $N$  and  $D$  are complex scalars,  $\zeta$ ,  $\tilde{\eta}$ ,  $\lambda$  and  $\tilde{\rho}$  are Weyl spinors and  $B_\mu$  is a *complex* vector field. The Hermitian conjugate,  $V^\dagger$ , is given by

$$V^\dagger(x, \theta, \tilde{\theta}) = C^*(x) + i\theta\zeta^c(x) + i\tilde{\theta}\tilde{\eta}^c(x) + \frac{1}{2}i\theta^2 M^*(x) + \frac{1}{2}i\tilde{\theta}^2 N^*(x) + \frac{1}{2}i\theta\sigma^\mu\tilde{\theta}B_\mu^*(x) - \frac{1}{2}\tilde{\theta}^2\theta\lambda^c(x) - \frac{1}{2}\theta^2\tilde{\theta}\tilde{\rho}^c(x) - \frac{1}{4}\theta^2\tilde{\theta}^2 D^*(x) , \quad (13)$$

where the relations (A.31) and (A.32) of Appendix A have been used.

The field-strength superfields,  $W_\alpha$  and  $\tilde{W}_{\dot{\alpha}}$  that satisfy respectively, the chiral and anti-chiral conditions  $\tilde{D}_{\dot{\beta}}W_\alpha = 0$  and  $D_\beta\tilde{W}_{\dot{\alpha}} = 0$ , are written as

$$W_\alpha = \frac{1}{2}\tilde{D}^2 D_\alpha V \quad \text{and} \quad \tilde{W}_{\dot{\alpha}} = \frac{1}{2}D^2 \tilde{D}_{\dot{\alpha}} V ; \quad (14)$$

in components we find

$$\begin{cases} W_\alpha = e^{i\tilde{\theta}\tilde{\theta}\theta} \left[ \hat{\lambda}_\alpha + \theta^\beta \left( \epsilon_{\alpha\beta} \widehat{D} - \sigma_{\alpha\beta}^{\mu\nu} G_{\mu\nu} \right) + i\theta^2 \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \hat{\rho}^{\dot{\alpha}} \right] \\ \tilde{W}_{\dot{\alpha}} = e^{i\theta\theta\tilde{\theta}} \left[ \hat{\rho}_{\dot{\alpha}} + \tilde{\theta}^{\dot{\beta}} \left( \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \widehat{D} - \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} G_{\mu\nu} \right) + i\tilde{\theta}^2 \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \partial_\mu \hat{\lambda}^\alpha \right] \end{cases} , \quad (15)$$

where

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha - \sigma_{\alpha}^{\mu\dot{\alpha}} \partial_\mu \tilde{\eta}_{\dot{\alpha}} \\ \widehat{D} = D - \square C \\ \hat{\rho}_{\dot{\alpha}} = \tilde{\rho}_{\dot{\alpha}} - \tilde{\sigma}_{\dot{\alpha}}^{\mu\alpha} \partial_\mu \zeta_\alpha \\ G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \end{cases} . \quad (16)$$

The supersymmetry transformation law (2) applied to  $W_\alpha$  and  $\tilde{W}_{\dot{\alpha}}$  yields the following changes for its component fields :

$$\begin{cases} \delta \hat{\lambda}_\alpha = \varepsilon^\beta \left( \epsilon_{\alpha\beta} \widehat{D} - \sigma_{\alpha\beta}^{\mu\nu} G_{\mu\nu} \right) \\ \delta \widehat{D} = i\varepsilon^\alpha \sigma_{\alpha}^{\mu\dot{\alpha}} \partial_\mu \hat{\rho}_{\dot{\alpha}} + i\tilde{\varepsilon}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}}^{\mu\alpha} \partial_\mu \hat{\lambda}_\alpha \\ \delta G_{\mu\nu} = -i\varepsilon^\alpha \sigma_{\mu\alpha}^{\dot{\alpha}} \partial_\nu \hat{\rho}_{\dot{\alpha}} + i\tilde{\varepsilon}^{\dot{\alpha}} \tilde{\sigma}_{\mu\dot{\alpha}}^{\alpha} \partial_\nu \hat{\lambda}_\alpha - (\mu \leftrightarrow \nu) \\ \delta \hat{\rho}_{\dot{\alpha}} = \tilde{\varepsilon}^{\dot{\beta}} \left( \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \widehat{D} - \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} G_{\mu\nu} \right) \end{cases} . \quad (17)$$

The charge conjugation of the field-strength superfields may be found as

$$\begin{cases} W_\alpha^c = e^{i\tilde{\theta}\tilde{\theta}\theta} \left[ \hat{\lambda}_\alpha + \theta^\beta \left( \epsilon_{\alpha\beta} \widehat{D}^* - \sigma_{\alpha\beta}^{\mu\nu} G_{\mu\nu}^* \right) + i\theta^2 \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \widehat{\rho}^{c\dot{\alpha}} \right] \\ \widetilde{W}_\alpha^c = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ \widehat{\rho}_\alpha^c + \tilde{\theta}^{\dot{\beta}} \left( \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \widehat{D}^* - \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} G_{\mu\nu}^* \right) + i\tilde{\theta}^2 \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \partial_\mu \widehat{\lambda}^{c\alpha} \right] \end{cases}, \quad (18)$$

where the relations (A.26), (A.31) and (A.32) are used. The connection between the complex conjugation and the charge conjugation of spinors plays an important rôle in the formulation of the supersymmetric gauge invariant actions as will be seen in the following sections.

### 3 Massive Abelian $N=1$ super-QED<sub>2+2</sub>

The supersymmetric extension of the massive Abelian QED in  $D=1+3$  requires two chiral superfields carrying opposite  $U(1)$ -charges [21]. On the other hand, to introduce mass for the matter sector in  $D=2+2$ , without breaking gauge-symmetry, we have to deal with four scalar superfields: a pair of chiral and a pair of anti-chiral supermultiplets; the members of each pair have opposite  $U(1)$ -charges.

The massive Abelian  $N=1$  super-QED<sub>2+2</sub> is described by the action : <sup>2</sup>

$$\begin{aligned} S_{\text{inv}}^{\text{AW}} = & -\frac{1}{8} \left( \int ds W^c W + \int d\tilde{s} \widetilde{W}^c \widetilde{W} \right) + \int dv \left( \Psi_+^\dagger e^{4qV} \widetilde{X}_+ + \Psi_-^\dagger e^{-4qV} \widetilde{X}_- \right) + \\ & + i m \left( \int ds \Psi_+ \Psi_- - \int d\tilde{s} \widetilde{X}_+ \widetilde{X}_- \right) + \text{h.c.} \quad , \end{aligned} \quad (19)$$

where  $q$  is a dimensionless coupling constant and  $m$  is a parameter with dimension of mass. The  $+$  and  $-$  subscripts in the matter superfields refer to their respective  $U(1)$ -charges. To build up the interaction terms, we have used a mixing between the chiral and anti-chiral superfields (in order to justify such a procedure, we refer to the works of Gates, Ketov and Nishino [6]). This mixed interaction term establishes that the vector superfield must be *complex*.

The gauge-fixing action in superspace is given by :

$$S_{\text{gf}}^{\text{AW}} = -\frac{1}{4\alpha} \int dv \left( \widetilde{D}^2 V^\dagger \right) \left( D^2 V \right) + \text{h.c.} \quad , \quad (20)$$

where  $\alpha$  is the dimensionless gauge-fixing parameter. It is worthwhile to specify that the superfield  $V$  and its corresponding field-strength superfields,  $W$ ,  $W^c$ ,  $\widetilde{W}$  and  $\widetilde{W}^c$ , are the same as the ones given in Section 2.

In the massive super-QED<sub>2+2</sub>-action, given by eq.(19), the chiral superfields  $\Psi_+$  and  $\Psi_-$  ( $\widetilde{D}_\alpha \Psi_\pm=0$ ), are defined as follows :

$$\Psi_\pm(x, \theta, \tilde{\theta}) = e^{i\tilde{\theta}\tilde{\theta}\theta} \left[ A_\pm(x) + i\theta\psi_\pm(x) + i\theta^2 F_\pm(x) \right] \quad , \quad (21)$$

---

<sup>2</sup>In this paper we are adopting  $ds \equiv d^4 x d^2 \theta$ ,  $d\tilde{s} \equiv d^4 x d^2 \tilde{\theta}$  and  $dv \equiv d^4 x d^2 \theta d^2 \tilde{\theta}$  for the superspace measures.

where  $A_{\pm}$  are complex scalars,  $\psi_{\pm}$  are Weyl spinors, and  $F_{\pm}$  are complex scalar auxiliary fields. Moreover, the anti-chiral superfields,  $\widetilde{X}_+$  and  $\widetilde{X}_-$  ( $D_{\alpha}\widetilde{X}_{\pm}=0$ ), are defined by :

$$\widetilde{X}_{\pm}(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ B_{\pm}(x) + i\theta\tilde{\chi}_{\pm}(x) + i\tilde{\theta}^2 G_{\pm}(x) \right] , \quad (22)$$

where  $B_{\pm}$  are complex scalars,  $\tilde{\chi}_{\pm}$  are Weyl spinors and,  $G_{\pm}$  are complex scalar auxiliary fields.

The transformations of the scalar superfields,  $\Psi_{\pm}$  and  $\widetilde{X}_{\pm}$ , that ensures the gauge invariance of the massive super-QED<sub>2+2</sub>-action (19) are the following:

$$\Psi_{\pm} \longrightarrow e^{\mp i4q\Lambda_{\pm}} \Psi_{\pm} , \quad \widetilde{D}_{\dot{\alpha}}\Lambda_{\pm} = 0 \quad \text{and} \quad \widetilde{X}_{\pm} \longrightarrow e^{\mp i4q\tilde{\Gamma}_{\pm}} \widetilde{X}_{\pm} , \quad D_{\alpha}\tilde{\Gamma}_{\pm} = 0 , \quad (23)$$

with

$$\Lambda_{\pm}(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ \Lambda_{1\pm}(x) + i\theta\Lambda_{2\pm}(x) + i\theta^2\Lambda_{3\pm}(x) \right] \quad (24)$$

and

$$\tilde{\Gamma}_{\pm}(x, \theta, \tilde{\theta}) = e^{i\theta\tilde{\theta}\tilde{\theta}} \left[ \Gamma_{1\pm}(x) + i\theta\tilde{\Gamma}_{2\pm}(x) + i\theta^2\Gamma_{3\pm}(x) \right] , \quad (25)$$

where  $\Lambda_{1\pm}$  and  $\Gamma_{1\pm}$  are complex scalars,  $\Lambda_{2\pm}$  and  $\tilde{\Gamma}_{2\pm}$  are Weyl spinors and,  $\Lambda_{3\pm}$  and  $\Gamma_{3\pm}$  are complex scalar auxiliary fields.

Taking into account the gauge invariance of the action (19) and assuming the transformations of the superfields  $\Psi_{\pm}$  and  $\widetilde{X}_{\pm}$  (23), it may be directly found that the superfield  $V$  suffers the following gauge transformations

$$\delta_g V = i \left( \tilde{\Gamma}_+ - \Lambda_+^{\dagger} \right) = i \left( \tilde{\Gamma}_- - \Lambda_-^{\dagger} \right) , \quad (26)$$

which puts in evidence the necessity for a *complex* vector superfield,  $V$ , in order to make possible the construction of a gauge-invariant action in the Atiyah-Ward space-time.

In terms of the component fields, the gauge transformation (26), by considering the eqs. (12), (24) and (25), becomes

$$\left\{ \begin{array}{l} \delta_g C = i \left( \Gamma_{1\pm} - \Lambda_{1\pm}^* \right) \\ \delta_g \zeta_{\alpha} = -i\Lambda_{2\pm\alpha}^c , \quad \delta_g \tilde{\eta}_{\dot{\alpha}} = i\tilde{\Gamma}_{2\pm\dot{\alpha}} \\ \delta_g M = -2i\Lambda_{3\pm}^* , \quad \delta_g N = 2i\Gamma_{3\pm} \\ \delta_g B_{\mu} = 2i\partial_{\mu}(\Gamma_{1\pm} + \Lambda_{1\pm}^*) \\ \delta_g \lambda_{\alpha} = -i\sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu} \tilde{\Gamma}_{2\pm\dot{\alpha}} , \quad \delta_g \tilde{\rho}_{\dot{\alpha}} = i\tilde{\sigma}_{\dot{\alpha}\alpha}^{\mu} \partial_{\mu} \Lambda_{2\pm\alpha}^c \\ \delta_g D = i\Box \left( \Gamma_{1\pm} - \Lambda_{1\pm}^* \right) \end{array} \right. . \quad (27)$$

Meanwhile, assuming the Wess-Zumino gauge [20, 21], by fixing the following non-supersymmetric gauge transformations :

$$\left\{ \begin{array}{l} \delta_g C = -C = i \left( \Gamma_{1\pm} - \Lambda_{1\pm}^* \right) \\ \delta_g \zeta_\alpha = -\zeta_\alpha = -i\Lambda_{2\pm\alpha}^c, \quad \delta_g \tilde{\eta}_{\dot{\alpha}} = -\tilde{\eta}_{\dot{\alpha}} = i\tilde{\Gamma}_{2\pm\dot{\alpha}} \\ \delta_g M = -M = -2i\Lambda_{3\pm}^*, \quad \delta_g N = -N = 2i\Gamma_{3\pm} \end{array} \right. . \quad (28)$$

By eliminating the compensating fields of the multiplet  $V$ , one is led to the following remaining transformation in the Wess-Zumino gauge :

$$\delta_g B_\mu = i \partial_\mu \beta, \quad (29)$$

where  $\beta$  is an arbitrary complex function. However, as we shall see below, after analysing the complete action in the Wess-Zumino gauge, the matter-gauge couplings indicate that indeed only the imaginary part of  $B_\mu$  displays the transformation of a genuine gauge field, whereas the real part of  $B_\mu$ -field gauges a Weyl symmetry. At this point, it should be mentioned that from now on, it will be omitted the hat ( $\wedge$ ) symbol over the components fields,  $\lambda$ ,  $D$  and  $\tilde{\eta}$ , since the calculations will always be performed in the Wess-Zumino gauge.

Adopting the Wess-Zumino gauge, the following component-field action stems from the superspace action of eq.(19) :

$$\begin{aligned} S_{\text{inv}}^{\text{AW}} = \int d^4x \left\{ -\frac{1}{4}i \left( \lambda^c \not{\partial} \tilde{\rho} + \tilde{\rho}^c \not{\partial} \lambda \right) - \frac{1}{8} G_{\mu\nu}^* G^{\mu\nu} - \frac{1}{4} D^* D + \right. \\ \left. -F_+^* G_+ - A_+^* \square B_+ - \frac{1}{2} i \psi_+^c \not{\partial} \tilde{\chi}_+ - q B_\mu \left( \frac{1}{2} i \psi_+^c \sigma^\mu \tilde{\chi}_+ + A_+^* \partial^\mu B_+ - B_+ \partial^\mu A_+^* \right) + \right. \\ \left. + i q \left( A_+^* \tilde{\chi}_+ \tilde{\rho} + B_+ \psi_+^c \lambda \right) - \left( q D + q^2 B_\mu B^\mu \right) A_+^* B_+ + \right. \\ \left. -F_-^* G_- - A_-^* \square B_- - \frac{1}{2} i \psi_-^c \not{\partial} \tilde{\chi}_- + q B_\mu \left( \frac{1}{2} i \psi_-^c \sigma^\mu \tilde{\chi}_- + A_-^* \partial^\mu B_- - B_- \partial^\mu A_-^* \right) + \right. \\ \left. - i q \left( A_-^* \tilde{\chi}_- \tilde{\rho} + B_- \psi_-^c \lambda \right) + \left( q D - q^2 B_\mu B^\mu \right) A_-^* B_- + \right. \\ \left. + m \left( \frac{1}{2} i \psi_+ \psi_- - \frac{1}{2} i \tilde{\chi}_+ \tilde{\chi}_- - A_+ F_- - A_- F_+ + B_+ G_- + B_- G_+ \right) \right\} + \text{h.c.} \quad (30) \end{aligned}$$

Also, in the Wess-Zumino gauge, the gauge-fixing action (20) is given in components by

$$S_{\text{gf}}^{\text{AW}} = \frac{1}{4\alpha} \int d^4x \left\{ -i \left( \lambda^c \not{\partial} \tilde{\rho} + \tilde{\rho}^c \not{\partial} \lambda \right) + \left( \partial^\mu B_\mu^* \right) \left( \partial^\nu B_\nu \right) - D^* D \right\} + \text{h.c.} \quad (31)$$

Due to the fact that in massive super-QED<sub>2+2</sub> one must have two opposite  $U(1)$ -charges to introduce mass at tree-level, and a complex vector superfield in order to build up the

gauge invariant interactions, we can read directly from the action (19), and the superfields, (12), (21) and (22), the following set of local  $U(1)_\alpha \times U(1)_\gamma$  transformations :

$$\left\{ \begin{array}{l} \delta_g A_\pm^* = \pm i q \beta(x) A_\pm^* \\ \delta_g \psi_\pm^c = \pm i q \beta(x) \psi_\pm^c \\ \delta_g F_\pm^* = \pm i q \beta(x) F_\pm^* \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \delta_g B_\pm = \mp i q \beta(x) B_\pm \\ \delta_g \tilde{\chi}_\pm = \mp i q \beta(x) \tilde{\chi}_\pm \\ \delta_g G_\pm = \mp i q \beta(x) G_\pm \end{array} \right. , \quad (32)$$

where  $\beta \equiv \alpha - i\gamma$  is an arbitrary infinitesimal  $C^\infty$  complex function. Notice that the gauge transformations (32) read as above because one has previously fixed to work in the Wess-Zumino gauge. As for the gauge superfield components surviving the Wess-Zumino gauge, we have :

$$\left\{ \begin{array}{l} \delta_g \lambda = \delta_g \tilde{\rho} = 0 \\ \delta_g D = 0 \\ \delta_g B_\mu = i \partial_\mu \beta \end{array} \right. \quad \text{and} \quad (33)$$

Therefore, in the Wess-Zumino gauge, the real part of  $B_\mu$  gauges the  $U(1)_\gamma$ -symmetry with real gauge function  $\gamma$ , whereas its imaginary part gauges the  $U(1)_\alpha$ -symmetry with real gauge function  $\alpha$ . The latter is an ordinary phase symmetry, and we associate it with the electric charge. Indeed, as we will see later on, the imaginary component of  $B_\mu$  will be taken as the photon field. The parameter  $\gamma$  generates a local Weyl-like invariance [22]. However, the vector field that gauges such a symmetry, namely the real part of  $B_\mu$ , will be suppressed in the process of dimensional reduction, so that such an invariance will not leave track in  $D=1+2$ .

It should be emphasized that the mass bilinears in the action given by eq.(30) preserve the local  $U(1)_\alpha \times U(1)_\gamma$ -symmetry, since their component matter fields (fermions and scalars) carry opposite charges. Therefore, the opposite values of the  $U(1)$ -charges play a central rôle in the process introducing mass for the matter fields without breakdown of the gauge-symmetry, similarly to what happens in  $D=1+3$ .

## 4 $N=1$ super- $\tau_3$ QED from Atiyah-Ward space-time

It is well-known that outstanding supersymmetric models with extended supersymmetry are closely related to simple ones in higher dimensions [14, 15]. As we are interested in simple supersymmetric models in  $D=1+2$ , since these ones should be fruitful for applications in Condensed Matter Physics [10], we propose here to investigate what kind of model comes out after a suitable compactification is adopted to dimensionally reduce Atiyah-Ward space-time to 3 space-time dimensions. Our propose is to carry out a dimensional reduction<sup>3</sup> of

---

<sup>3</sup>One uses the trivial dimensional reduction where the time-derivative,  $\partial_3$ , of all component fields vanishes,  $\partial_3 \mathcal{F} = 0$ . See also the Appendix B.

the massive  $N=1$  super-QED<sub>2+2</sub> *à la* Scherk [14]. Bearing in mind that this procedure should extend the supersymmetry [14, 15] to  $N>1$ , truncations will be needed in order to remain with a simple supersymmetry and to suppress non-physical modes, *i.e.* spurious degrees of freedom coming from  $D=2+2$  dimensions.

To perform the dimensional reduction *à la* Scherk from  $D=2+2$  to  $D=1+2$  of the action (30), use has been made of the rules presented in the Appendix B (see (B.28)–(B.37)). As a result, it can be directly found the following supersymmetric action in  $D=1+2$  :

$$\begin{aligned}
S_{\text{inv}}^{D=3} = & \int d^3\hat{x} \left\{ -\frac{1}{4}i \left( \bar{\lambda}\gamma^m\partial_m\rho + \bar{\rho}\gamma^m\partial_m\lambda \right) - \frac{1}{8} \left( G_{mn}^* G^{mn} + 2\partial_m\phi^*\partial^m\phi \right) - \frac{1}{4}D^*D + \right. \\
& -F_+^*G_+ - A_+^*\square B_+ - \frac{1}{2}i\bar{\psi}_+\gamma^m\partial_m\chi_+ - qB_m \left( \frac{1}{2}i\bar{\psi}_+\gamma^m\chi_+ + A_+^*\partial^m B_+ - B_+\partial^m A_+^* \right) + \\
& + \frac{1}{2}q\phi\bar{\psi}_+\chi_+ + q \left( A_+^*\bar{\chi}_+\rho - B_+\bar{\psi}_+\lambda \right) - \left( qD + q^2B_mB^m + q^2\phi^2 \right) A_+^*B_+ + \\
& -F_-^*G_- - A_-^*\square B_- - \frac{1}{2}i\bar{\psi}_-\gamma^m\partial_m\chi_- + qB_m \left( \frac{1}{2}i\bar{\psi}_-\gamma^m\chi_- + A_-^*\partial^m B_- - B_-\partial^m A_-^* \right) + \\
& - \frac{1}{2}q\phi\bar{\psi}_-\chi_- - q \left( A_-^*\bar{\chi}_-\rho - B_-\bar{\psi}_-\lambda \right) + \left( qD - q^2B_mB^m - q^2\phi^2 \right) A_-^*B_- + \\
& \left. -m \left( \frac{1}{2}\bar{\psi}_+\psi_- + \frac{1}{2}\bar{\chi}_+\chi_- + A_+F_- + A_-F_+ - B_+G_- - B_-G_+ \right) \right\} + \text{h.c.} \quad , \quad (34)
\end{aligned}$$

where, after dimensional reduction, the coupling constant  $q$  has acquired dimension of  $(\text{mass})^{\frac{1}{2}}$ . Furthermore, after performing the dimensional reduction of the gauge-fixing (31), we found the following gauge-fixing action in  $D=1+2$  :

$$S_{\text{gf}}^{D=3} = \frac{1}{4\alpha} \int d^3\hat{x} \left\{ -i \left( \bar{\lambda}\gamma^m\partial_m\rho + \bar{\rho}\gamma^m\partial_m\lambda \right) - (\partial^m B_m^*)(\partial^n B_n) - D^*D \right\} + \text{h.c.} \quad . \quad (35)$$

Analysing the 3-dimensional action<sup>4</sup> given by eq.(34), it can be easily shown that the spectrum will unavoidably be spoiled by the presence of ghost fields, since the free sector of the action is totally off-diagonal. Therefore, truncations are needed in order to remove the spurious degrees of freedom, as well as to give rise to a simple supersymmetric action in  $D=1+2$ . First of all, to make the truncations possible, we need to diagonalize the whole free sector, in order that the ghost fields be identified.

In order to probe more deeply such a conclusion, we should diagonalize the free gauge and matter sectors of the action (34). The diagonalization is achieved by looking for suitable linear combinations of the fields which yield a diagonal free action. After tedious algebraic manipulations<sup>5</sup>, we find the following transformations which diagonalize the action  $S_{\text{inv}}^{D=3}$  :

1. gauge sector :

$$\lambda = \frac{1}{\sqrt{2}} (\hat{\rho} + \hat{\lambda}) \quad \text{and} \quad \rho = \frac{1}{\sqrt{2}} (\hat{\rho} - \hat{\lambda}) \quad ; \quad (36)$$

---

<sup>4</sup>Note that,  $\lambda$ ,  $\rho$ ,  $\psi$  and  $\chi$  are now Dirac spinors in  $D=1+2$ .

<sup>5</sup>For the calculations, use has been made of the **MapleV** software, since, to fix the appropriate parameters present in the massive Abelian  $N=1$  super-QED<sub>2+2</sub> action (19) so as to get (34), we had to invert and diagonalize  $8\times 8$  and  $16\times 16$  matrices.

2. fermionic matter sector :

$$\psi_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{\psi}_{\pm} \mp \hat{\psi}_{\mp}^c + \hat{\chi}_{\pm} \pm \hat{\chi}_{\mp}^c \right) \quad \text{and} \quad \chi_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{\chi}_{\pm} \pm \hat{\chi}_{\mp}^c - \hat{\psi}_{\pm} \pm \hat{\psi}_{\mp}^c \right) ; \quad (37)$$

3. bosonic matter sector :

$$A_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{A}_{\pm} - \hat{B}_{\pm} \right) \quad \text{and} \quad B_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{A}_{\pm} + \hat{B}_{\pm} \right) ; \quad (38)$$

$$F_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{F}_{\pm} + \hat{G}_{\pm} \right) \quad \text{and} \quad G_{\pm} = \frac{1}{\sqrt{2}} \left( \hat{G}_{\pm} - \hat{F}_{\pm} \right) . \quad (39)$$

On the other hand, to simplify the Yukawa-interaction terms (gaugino-matter couplings), we find that the following field redefinitions for the bosonic matter sector are convenient :

$$\hat{A}_{\pm} = \frac{1}{\sqrt{2}} \left( \check{A}_{\pm} \mp \check{A}_{\mp}^* \right) \quad \text{and} \quad \hat{F}_{\pm} = \frac{1}{\sqrt{2}} \left( \check{F}_{\pm} \mp \check{F}_{\mp}^* \right) . \quad (40)$$

By replacing these field redefinitions into the action (34), one ends up with a diagonalized action, where the fields,  $\phi$ ,  $\hat{\rho}$ ,  $\hat{\chi}_+$ ,  $\hat{\chi}_-$ ,  $\hat{B}_+$  and  $\hat{B}_-$  appear like ghosts in the framework of an  $N=2$ -supersymmetric model. Therefore, in order to suppress these non-physical modes, truncations must be performed. Bearing in mind that we are looking for an  $N=1$  supersymmetric 3-dimensional model (in the Wess-Zumino gauge), truncations have to be imposed on the ghost fields,  $\phi$ ,  $\hat{\rho}$ ,  $\hat{\chi}_+$ ,  $\hat{\chi}_-$ ,  $\hat{B}_+$  and  $\hat{B}_-$ . To keep  $N=1$  supersymmetry in the Wess-Zumino gauge, we must simultaneously truncate the component fields,  $\hat{G}_+$ ,  $\hat{G}_-$ ,  $D$ ,  $a_m$  and  $\tau$ <sup>6</sup>. The truncation of  $\tau$  is dictated by the suppression of  $a_m$ . Now, the choice of truncating  $a_m$ , instead of  $A_m$ , is based on the analysis of the couplings to the matter sector:  $A_m$  couples to both scalar and fermionic matter and we interpret it as the photon field in 3 dimensions.

After these truncations are performed, and omitting the  $(\wedge)$  and  $(\vee)$  symbols, we find the following action in  $D=1+2$  :

$$\begin{aligned} S_{N=1}^{\tau_3 \text{QED}} = \int d^3x \left\{ \frac{1}{2} i \bar{\lambda} \gamma^m \partial_m \lambda - \frac{1}{4} F_{mn} F^{mn} + \right. \\ - A_+^* \square A_+ - A_-^* \square A_- + i \bar{\psi}_+ \gamma^m \partial_m \psi_+ + i \bar{\psi}_- \gamma^m \partial_m \psi_- + F_+^* F_+ + F_-^* F_- + \\ - q A_m \left( \bar{\psi}_+ \gamma^m \psi_+ - \bar{\psi}_- \gamma^m \psi_- + i A_+^* \partial^m A_+ - i A_-^* \partial^m A_- - i A_+ \partial^m A_+^* + i A_- \partial^m A_-^* \right) + \\ - i q \left( A_+ \bar{\psi}_+ \lambda - A_- \bar{\psi}_- \lambda - A_+^* \bar{\lambda} \psi_+ + A_-^* \bar{\lambda} \psi_- \right) + q^2 A_m A^m \left( A_+^* A_+ + A_-^* A_- \right) + \\ \left. - m \left( \bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_- + A_+^* F_+ - A_-^* F_- + A_+ F_+^* - A_- F_-^* \right) \right\} , \quad (41) \end{aligned}$$

where it can be readily concluded that this is a supersymmetric extension of a parity-preserving action, namely,  $\tau_3 \text{QED}$  [12]. However, to render our claim more explicit, we

---

<sup>6</sup>The  $a_m$  field is the real part of  $B_m$ , since we are assuming  $B_m = a_m + i A_m$ . Also, as  $\hat{\lambda}$  is a Dirac spinor, it can be written in terms of two Majorana spinors in the following manner:  $\hat{\lambda} = \tau + i \lambda$ .

are going to make use of a superspace formulation, where the superfields are conveniently defined and the notational conventions are fixed by the dimensional reduction. Before that, it should be relevant to show how the gauge-fixing action (35) appears after these suitable truncations :

$$S_{\text{gf}}^{\tau_3 \text{QED}} = \frac{1}{2\alpha} \int d^3\hat{x} \left\{ i\bar{\lambda}\gamma^m \partial_m \lambda + (\partial^m A_m)^2 \right\} \quad . \quad (42)$$

In order to formulate the  $N=1$  super- $\tau_3$ QED action (41) in terms of superfields, we refer to the work by Salam and Strathdee [19], where the superspace and superfields in  $D=1+3$  were formulated for the first time. Extending their ideas to our case in  $D=1+2$  (see also ref.[17]), the elements of superspace are labeled by  $(x^m, \theta)$ , where  $x^m$  are the space-time coordinates and the fermionic coordinates,  $\theta$ , are Majorana spinors,  $\theta^c = \theta$ .<sup>7</sup>

Now, we are ready to introduce the formulation of  $N=1$  super- $\tau_3$ QED in terms of superfields. As a first step, we define the complex scalar superfields with opposite  $U(1)$ -charges,  $\Phi_+$  and  $\Phi_-$ , as

$$\Phi_{\pm} = A_{\pm} + \bar{\theta}\psi_{\pm} - \frac{1}{2}\bar{\theta}\theta F_{\pm} \quad \text{and} \quad \Phi_{\pm}^{\dagger} = A_{\pm}^* + \bar{\psi}_{\pm}\theta - \frac{1}{2}\bar{\theta}\theta F_{\pm}^* \quad , \quad (43)$$

where  $A_{\pm}$  are complex scalars,  $\psi_{\pm}$  are Dirac spinors and  $F_{\pm}$  are complex scalar auxiliary fields.

In the Wess-Zumino gauge, the gauge superconnection,  $\Gamma_a$ , is written as

$$\Gamma_a = i(\gamma^m \theta)_a A_m + \bar{\theta}\theta \lambda_a \quad \text{and} \quad \bar{\Gamma}_a = -i(\bar{\theta}\gamma^m)_a A_m + \bar{\theta}\theta \bar{\lambda}_a \quad , \quad (44)$$

where  $A_m$  is the gauge field and  $\lambda_a$  is the gaugino (Majorana spinor).

Defining the field-strength superfield,  $W_a$ , according to :

$$W_a = -\frac{1}{2}\bar{D}_b D_a \Gamma_b \quad , \quad (45)$$

with superderivatives given by

$$D_a = \bar{\partial}_a - i(\gamma^m \theta)_a \partial_m \quad \text{and} \quad \bar{D}_a = -\partial_a + i(\bar{\theta}\gamma^m)_a \partial_m \quad , \quad (46)$$

it can be found that

$$W_a = \lambda_a + \Sigma^{mn}_{ab} \theta_b F_{mn} - \frac{i}{2}\bar{\theta}\theta \gamma^m_{ab} (\partial_m \lambda_b) \quad (47.a)$$

and

$$\bar{W}_a = \bar{\lambda}_a - \bar{\theta}_b \Sigma^{mn}_{ba} F_{mn} + \frac{i}{2}\bar{\theta}\theta (\partial_m \bar{\lambda}_b) \gamma^m_{ba} \quad , \quad (47.b)$$

where  $\Sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n]$  are the generators of the Lorentz group in  $D=1+2$ .

---

<sup>7</sup>The charge-conjugated spinor is defined by  $\psi^c = -C\bar{\psi}^T$ , where  $C = \sigma_y$ . The  $\gamma$ -matrices we are using arised from the dimensional reduction to  $D=1+2$  are:  $\gamma^m = (\sigma_x, i\sigma_y, -i\sigma_z)$ . Note that for any spinorial objects,  $\psi$  and  $\chi$ , the product  $\bar{\psi}\chi$  denotes  $\bar{\psi}_a \chi_a$ . For more details, see the Appendix B.

The gauge covariant derivatives we are defining for the matter superfields with opposite  $U(1)$ -charges,  $\Phi_+$  and  $\Phi_-$ , are given by

$$\nabla_a \Phi_{\pm} = (D_a \mp iq\Gamma_a) \Phi_{\pm} \quad \text{and} \quad \bar{\nabla}_a \Phi_{\pm}^{\dagger} = (\bar{D}_a \pm iq\bar{\Gamma}_a) \Phi_{\pm}^{\dagger} \quad , \quad (48)$$

where  $q$  is a coupling constant with dimension of  $(\text{mass})^{\frac{1}{2}}$ .

By using the previous definitions of the superfields, (43), (44), (47.a)-(47.b), and the gauge covariant derivatives, (48), we found how to build up the  $N=1$  super- $\tau_3$ QED action, given by eq.(41), in superspace ; it reads :

$$S_{N=1}^{\tau_3\text{QED}} = \int d\hat{v} \left\{ -\frac{1}{2} \bar{W}W + (\bar{\nabla}\Phi_+^{\dagger})(\nabla\Phi_+) + (\bar{\nabla}\Phi_-^{\dagger})(\nabla\Phi_-) + 2m(\Phi_+^{\dagger}\Phi_+ - \Phi_-^{\dagger}\Phi_-) \right\} \quad , \quad (49)$$

where the superspace measure we have adopted is  $d\hat{v} \equiv d^3\hat{x}d^2\theta$  and the Berezin integral is taken as  $\int d^2\theta = -\frac{1}{4}\bar{\partial}\partial$  (see the Appendix B). Therefore, we finally show, by using the superspace formulation (49), that the action (41) we have found after a dimensional reduction *à la* Scherk, and some suitable truncations of the massive Abelian  $N=1$  super-QED $_{2+2}$ , is certainly the simple supersymmetric version of  $\tau_3$ QED.

The gauge-fixing action (42) can be written in superspace as

$$S_{\text{gf}}^{\tau_3\text{QED}} = -\frac{1}{8\alpha} \int d\hat{v} \left\{ (\bar{D}\Gamma) \bar{D}D (\bar{D}\Gamma) \right\} \quad . \quad (50)$$

We conclude this section by pointing out that the massive Abelian  $N=1$  super-QED $_{2+2}$  proposed in Section 3 shows interesting features, whenever an appropriate dimensional reduction is performed. The dimensional reduction *à la* Scherk we have applied to our problem becomes very attractive, since, after doing some truncations to avoid non-physical modes, the  $N=1$  super- $\tau_3$ QED is obtained as a final result. In fact, the Atiyah-Ward space-time shows to be very fascinating as a starting point to formulate models to be studied in lower dimensions.

## 5 Discussions and general conclusions

We attempted here to provide a connection between the  $N=1$  supersymmetric version of the parity-preserving  $\tau_3$ QED in  $D=1+2$  and the minimal version of an  $N=1$  supersymmetric QED $_{2+2}$  in Atiyah-Ward space-time.

The superspace formulation of the model in  $D=2+2$  reveals the peculiar features of a complex gauge superconnection and an Abelian symmetry of the type  $U(1)_{\alpha} \times U(1)_{\gamma}$ , where a local Weyl-like symmetry is present. The reduction to  $D=1+2$  *à la* Scherk truncates, however, the gauge field associated to this symmetry. But, we think that a better understanding of such an invariance could be of relevance in connection with the formulation of a conformally-invariant  $N=1$  supergravity model coupled to the super-QED $_{2+2}$  studied here [23].

Space-time supersymmetry by itself is a good motivation to introduce self-dual theories which have a good chance to be the generating theories for all supersymmetric integrable

models in lower dimensions. We would like to point out that it would be worthwhile to study the possibility that our massive Abelian model gives rise to a self-dual gauge field. This can be done on the basis of the Parkes-Siegel formulation [24], which after carrying out a suitable dimensional reduction *à la* Nishino [7] to  $D=1+2$  generates a 3-dimensional model with a Chern-Simons term for the  $B_\mu$ -component of the vector supermultiplet [25].

The Parkes-Siegel formulation for the massive Abelian  $N=1$  super-QED $_{2+2}$  coupled to a self-dual supermultiplet is achieved by introducing a chiral multiplier superfield ( $\widetilde{D}_{\dot{\beta}}\Xi_\alpha = 0$ ); its action is given by

$$S_{\text{SQED}}^{\text{SD}} = - \int ds \, \Xi^c W + \int dv \, \left( \Psi_+^\dagger e^{4qV} \widetilde{X}_+ + \Psi_-^\dagger e^{-4qV} \widetilde{X}_- \right) + \\ + i m \left( \int ds \, \Psi_+ \Psi_- - \int d\tilde{s} \, \widetilde{X}_+ \widetilde{X}_- \right) + \text{h.c.} \quad , \quad (51)$$

with

$$\Xi_\alpha = e^{i\tilde{\theta}\tilde{\phi}\theta} \left[ A_\alpha + \theta^\beta \left( \epsilon_{\alpha\beta} E - \sigma_{\alpha\beta}^{\mu\nu} H_{\mu\nu} \right) + i\theta^2 F_\alpha \right] \quad , \quad (52)$$

where  $A_\alpha$  is a Weyl spinor,  $E$  is a complex scalar,  $H_{\mu\nu}$  is a complex antisymmetric rank-2 tensor and  $F_\alpha$  is a Weyl auxiliary spinor.

By adopting the Wess-Zumino gauge, the following component-field action stems from the superspace action of eq.(51) :

$$S_{\text{SQED}}^{\text{SD}} = \int d^4x \left\{ -\frac{1}{2} H_{\mu\nu}^* \left( G^{\mu\nu} - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} \right) - i \left( A^c \tilde{\phi} \tilde{\rho} + F^c \lambda \right) - E^* D + \right. \\ - F_+^* G_+ - A_+^* \square B_+ - \frac{1}{2} i \psi_+^c \tilde{\phi} \tilde{\chi}_+ - q B_\mu \left( \frac{1}{2} i \psi_+^c \sigma^\mu \tilde{\chi}_+ + A_+^* \partial^\mu B_+ - B_+ \partial^\mu A_+^* \right) + \\ + i q \left( A_+^* \tilde{\chi}_+ \tilde{\rho} + B_+ \psi_+^c \lambda \right) - \left( q D + q^2 B_\mu B^\mu \right) A_+^* B_+ + \\ - F_-^* G_- - A_-^* \square B_- - \frac{1}{2} i \psi_-^c \tilde{\phi} \tilde{\chi}_- + q B_\mu \left( \frac{1}{2} i \psi_-^c \sigma^\mu \tilde{\chi}_- + A_-^* \partial^\mu B_- - B_- \partial^\mu A_-^* \right) + \\ - i q \left( A_-^* \tilde{\chi}_- \tilde{\rho} + B_- \psi_-^c \lambda \right) + \left( q D - q^2 B_\mu B^\mu \right) A_-^* B_- + \\ \left. + m \left( \frac{1}{2} i \psi_+ \psi_- - \frac{1}{2} i \tilde{\chi}_+ \tilde{\chi}_- - A_+ F_- - A_- F_+ + B_+ G_- + B_- G_+ \right) \right\} + \text{h.c.} \quad . \quad (53)$$

Therefore, it can be easily seen by action (53), that the field equation of  $H_{\mu\nu}^*$  gives the self-duality of the field-strength,  $G^{\mu\nu}$  :

$$\frac{\delta S_{\text{SQED}}^{\text{SD}}}{\delta H_{\mu\nu}^*} = 0 \quad \implies \quad G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} \quad . \quad (54)$$

It would be a good suggestion to consider the dimensional reduction of the action (53) according to the prescription proposed by Nishino in ref.[7], and to try to understand the implications of the 3-dimensional theory in connection with the phenomenology of  $\tau_3\text{QED}_{1+2}$ . As a final remark, we point out that the super-Yang-Mills version of the work presented here could be of interest in association to the self-duality condition and the reduction of the model from  $D=2+2$  to  $D=1+1$ , in order to check which sort of integrable model may drop out from the reduction procedure.

## A General notations and conventions for $D=2+2$

We begin by reviewing some aspects of spinors living in the Atiyah-Ward space-time. The Dirac spinor,  $\Psi$ , for even dimensions, may be represented, by using the chiral operators, in terms of two Weyl spinors  $\psi$  and  $\tilde{\chi}$ . Each of the Weyl spinors transforms under the action of the group  $SL(2, \mathbb{R})$  [5]. In the Weyl representation the Dirac spinor takes the form :

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\chi} \end{pmatrix}, \quad (\text{A.1})$$

where  $\psi$  and  $\tilde{\chi}$  have the following components:  $\psi^\alpha$ ,  $\alpha=(1, 2)$ , and  $\tilde{\chi}^{\dot{\alpha}}$ ,  $\dot{\alpha}=(\dot{1}, \dot{2})$ .

The Dirac  $\gamma$ -matrices can be represented by  $4 \times 4$  complex matrices that satisfy the Clifford algebra :

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_4, \quad (\text{A.2})$$

where  $\mathbb{1}_4$  is the  $4 \times 4$  identity matrix. Since the matrices  $-\gamma^{\mu\dagger}$ ,  $\gamma^{\mu*}$  and  $-\gamma^{\mu T}$  obey the same Clifford algebra as the  $\gamma^\mu$ , and there is only one irreducible representation of the Clifford algebra by complex  $4 \times 4$  matrices up to equivalence transformations, there exist matrices  $A$ ,  $B$  and  $C$  with

$$\gamma^{\mu\dagger} = -A\gamma^\mu A^{-1}, \quad (\text{A.3})$$

$$\gamma^{\mu*} = B\gamma^\mu B^{-1}, \quad (\text{A.4})$$

$$\gamma^{\mu T} = -C\gamma^\mu C^{-1}, \quad (\text{A.5})$$

where  $A=\gamma^0\gamma^3$ . The matrix,  $B$ , and the charge conjugation matrix,  $C$ , in Weyl representation are given by

$$B = \begin{pmatrix} i\sigma_z & \mathbf{0} \\ \mathbf{0} & i\sigma_z \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \epsilon & \mathbf{0} \\ \mathbf{0} & \tilde{\epsilon} \end{pmatrix}, \quad (\text{A.6})$$

where

$$\epsilon = i\sigma_y \quad \text{and} \quad \tilde{\epsilon} = -i\sigma_y. \quad (\text{A.7})$$

The  $\gamma$ -matrices in this Weyl representation are written as :

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \tilde{\sigma}^\mu & \mathbf{0} \end{pmatrix}, \quad (\text{A.8})$$

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad (\text{A.9})$$

where  $\mathbb{1}_2$  is the  $2 \times 2$  identity matrix. Besides, the  $\sigma$ -matrices of (A.8) have the following components :

$$\sigma^\mu = (-i\sigma_x, \sigma_y, -\sigma_z, \mathbb{1}_2), \quad (\text{A.10})$$

$$\tilde{\sigma}^\mu = (i\sigma_x, -\sigma_y, \sigma_z, \mathbb{1}_2), \quad (\text{A.11})$$

where the usual Pauli matrices read

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.12})$$

The  $\overline{SO(2,2)}$ -group has the following generators in the spinorial representation :

$$\Sigma^{\kappa\lambda} = \frac{1}{4}[\gamma^\kappa, \gamma^\lambda] = \begin{pmatrix} \sigma^{\kappa\lambda} & \mathbf{0} \\ \mathbf{0} & \tilde{\sigma}^{\kappa\lambda} \end{pmatrix}. \quad (\text{A.13})$$

Therefore, by using the eqs. (A.8) and (A.13), the  $\sigma$  and  $\tilde{\sigma}$  matrices read

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu) \quad \text{and} \quad \tilde{\sigma}^{\mu\nu} = \frac{1}{4}(\tilde{\sigma}^\mu \sigma^\nu - \tilde{\sigma}^\nu \sigma^\mu). \quad (\text{A.14})$$

The complex conjugation of the matrices,  $\sigma^\mu$ ,  $\tilde{\sigma}^\mu$ ,  $\sigma^{\mu\nu}$  and  $\tilde{\sigma}^{\mu\nu}$  results

$$\sigma^{\mu*} = \sigma_z \sigma^\mu \sigma_z \quad \text{and} \quad \tilde{\sigma}^{\mu*} = \sigma_z \tilde{\sigma}^\mu \sigma_z; \quad (\text{A.15})$$

$$\sigma^{\mu\nu*} = \sigma_z \sigma^{\mu\nu} \sigma_z \quad \text{and} \quad \tilde{\sigma}^{\mu\nu*} = \sigma_z \tilde{\sigma}^{\mu\nu} \sigma_z. \quad (\text{A.16})$$

Other useful relations involving the  $\sigma$ -matrices and their traces (Tr) used in the calculations are given by :

$$\text{Tr}(\sigma^\mu \tilde{\sigma}^\nu) = 2\eta^{\mu\nu}, \quad (\text{A.17})$$

$$(\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu)^\alpha_\beta = 2\eta^{\mu\nu} \delta^\alpha_\beta, \quad (\text{A.18})$$

$$(\tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu)^\alpha_\beta = 2\eta^{\mu\nu} \delta^\alpha_\beta; \quad (\text{A.19})$$

and

$$\text{Tr}(\sigma^{\mu\nu} \sigma^{\kappa\lambda}) = \frac{1}{2}(\eta^{\mu\lambda} \eta^{\nu\kappa} - \eta^{\mu\kappa} \eta^{\nu\lambda} + \epsilon^{\mu\nu\kappa\lambda}), \quad (\text{A.20})$$

$$\text{Tr}(\tilde{\sigma}^{\mu\nu} \tilde{\sigma}^{\kappa\lambda}) = \frac{1}{2}(\eta^{\mu\lambda} \eta^{\nu\kappa} - \eta^{\mu\kappa} \eta^{\nu\lambda} - \epsilon^{\mu\nu\kappa\lambda}). \quad (\text{A.21})$$

The charge-conjugated spinor,  $\Psi^c$ , is defined as follows

$$\Psi^c = B\Psi^* = C\overline{\Psi}^T, \quad (\text{A.22})$$

with  $\overline{\Psi} = \Psi^\dagger A$ , where  $A = \gamma^0 \gamma^3$ . In the Weyl representation, the charge-conjugated spinor read as

$$\Psi^c = \begin{pmatrix} \psi^c \\ \tilde{\chi}^c \end{pmatrix}, \quad (\text{A.23})$$

where  $\psi^c \equiv i\sigma_z \psi^*$  and  $\tilde{\chi}^c \equiv i\sigma_z \tilde{\chi}^*$ . For the properties of charge-conjugated spinors living in  $D=t+s$  space-time dimensions see ref.[26].

The charge conjugation operation upon  $\Psi$  for  $D=2+2$  does not mix the chiral sectors, since the matrix  $B$  is diagonal in the Weyl representation. Bearing in mind that the covering group of  $SO(2, 2)$  has the well-known isomorphism  $\overline{SO(2, 2)} \cong SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})$  [27], it may be concluded that  $\psi^c$  and  $\tilde{\chi}^c$  transforms in the same manner as  $\psi$  and  $\tilde{\chi}$  respectively, where the Weyl conjugated-spinors  $\psi^c$  and  $\tilde{\chi}^c$  have the following components:  $\psi^{c\alpha}$  and  $\tilde{\chi}^{c\dot{\alpha}}$ .

The Majorana spinor is defined by the constraint  $\Psi = \Psi^c$ . Therefore, due to the fact that  $B$  is diagonal in the Weyl representation, it follows that in components we have  $\psi = \psi^c$  and  $\tilde{\chi} = \tilde{\chi}^c$ . The Weyl spinors which satisfy these constraints are called Majorana-Weyl spinors [5]. In the case of  $D=1+3$ , it is well-known that Majorana-Weyl spinors do not exist, since it is not possible to impose simultaneously the Majorana and Weyl conditions.

## Index conventions

For all  $\psi, \tilde{\chi}, \sigma^\mu, \tilde{\sigma}^\mu, \epsilon, \tilde{\epsilon}$  that appear in the text, we adopt the following conventions for the index structure :  $\psi^\alpha, \tilde{\chi}^{\dot{\alpha}}, \sigma^{\mu\alpha}_{\dot{\alpha}}, \tilde{\sigma}^{\mu\dot{\alpha}}_{\alpha}, \epsilon_{\alpha\beta}, \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}}$ . In addition to this, we consider the symbols  $\epsilon^{\alpha\beta}$  and  $\tilde{\epsilon}^{\dot{\alpha}\dot{\beta}}$ , such that  $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$  and  $\tilde{\epsilon}^{\dot{\alpha}\dot{\beta}}\tilde{\epsilon}_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}$ , which act on the two independent  $SL(2, \mathbb{R})$  sectors. Therefore, the spinor indices are raised and lowered according to the rules :

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \quad \text{and} \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad ; \quad (\text{A.24})$$

$$\tilde{\chi}_{\dot{\alpha}} = \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}}\tilde{\chi}^{\dot{\beta}} \quad \text{and} \quad \tilde{\chi}^{\dot{\alpha}} = \tilde{\epsilon}^{\dot{\alpha}\dot{\beta}}\tilde{\chi}_{\dot{\beta}} \quad . \quad (\text{A.25})$$

The charge-conjugated Weyl spinors are given by

$$\psi^{c\alpha} = (i\sigma_z\psi^*)^\alpha \quad , \quad \tilde{\chi}^{c\dot{\alpha}} = (i\sigma_z\tilde{\chi}^*)^{\dot{\alpha}} \quad . \quad (\text{A.26})$$

Some of the  $SL(2, \mathbb{R})$  invariant bilinears can be briefly written as

$$\psi^\alpha\psi_\alpha \equiv \psi^2 \quad \text{and} \quad \psi^\alpha\lambda_\alpha \equiv \psi\lambda = \lambda\psi \quad ; \quad (\text{A.27})$$

$$\tilde{\chi}^{\dot{\alpha}}\tilde{\chi}_{\dot{\alpha}} \equiv \tilde{\chi}^2 \quad \text{and} \quad \tilde{\chi}^{\dot{\alpha}}\tilde{\rho}_{\dot{\alpha}} \equiv \tilde{\chi}\tilde{\rho} = \tilde{\rho}\tilde{\chi} \quad . \quad (\text{A.28})$$

Also, other bilinears are such that

$$\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}} \equiv \psi\sigma^\mu\tilde{\chi} = -\tilde{\chi}\tilde{\sigma}^\mu\psi \quad , \quad (\text{A.29})$$

$$\tilde{\rho}^{\dot{\alpha}}\tilde{\sigma}^\mu_{\dot{\alpha}\alpha}\lambda^\alpha \equiv \tilde{\rho}\tilde{\sigma}^\mu\lambda = -\lambda\sigma^\mu\tilde{\rho} \quad . \quad (\text{A.30})$$

Their complex conjugation yields

$$(i\psi\lambda)^* = i\psi^c\lambda^c \quad , \quad (i\tilde{\chi}\tilde{\rho})^* = i\tilde{\chi}^c\tilde{\rho}^c \quad ; \quad (\text{A.31})$$

$$(i\psi\sigma^\mu\tilde{\chi})^* = i\psi^c\sigma^\mu\tilde{\chi}^c \quad , \quad (i\tilde{\rho}\tilde{\sigma}^\mu\lambda)^* = i\tilde{\rho}^c\tilde{\sigma}^\mu\lambda^c \quad . \quad (\text{A.32})$$

Some useful relations for the Majorana-Weyl spinors  $\theta$  and  $\tilde{\theta}$  follow :

$$\theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta^2 \quad , \quad (\text{A.33})$$

$$\theta_\alpha \theta_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \theta^2 \quad , \quad (\text{A.34})$$

$$\tilde{\theta}^{\dot{\alpha}} \tilde{\theta}^{\dot{\beta}} = -\frac{1}{2} \tilde{\epsilon}^{\dot{\alpha}\dot{\beta}} \tilde{\theta}^2 \quad , \quad (\text{A.35})$$

$$\tilde{\theta}_{\dot{\alpha}} \tilde{\theta}_{\dot{\beta}} = \frac{1}{2} \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \tilde{\theta}^2 \quad , \quad (\text{A.36})$$

$$\theta \sigma^\mu \tilde{\theta} \theta \sigma^\nu \tilde{\theta} = \frac{1}{2} \theta^2 \tilde{\theta}^2 \eta^{\mu\nu} \quad . \quad (\text{A.37})$$

The fermionic derivatives are defined as :

$$\partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} \quad , \quad (\text{A.38})$$

$$\tilde{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial \tilde{\theta}^{\dot{\alpha}}} \quad . \quad (\text{A.39})$$

Therefore, it follows that

$$\partial_\alpha \theta^\beta = \delta_\alpha^\beta \quad , \quad \tilde{\partial}_{\dot{\alpha}} \tilde{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \quad ; \quad (\text{A.40})$$

$$\partial^\alpha \theta_\beta = -\delta^\alpha_\beta \quad , \quad \tilde{\partial}^{\dot{\alpha}} \tilde{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}} \quad ; \quad (\text{A.41})$$

$$\partial_\alpha \theta_\beta = -\epsilon_{\alpha\beta} \quad , \quad \tilde{\partial}_{\dot{\alpha}} \tilde{\theta}_{\dot{\beta}} = -\tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \quad ; \quad (\text{A.42})$$

$$\partial^\alpha \theta^\beta = \epsilon^{\alpha\beta} \quad , \quad \tilde{\partial}^{\dot{\alpha}} \tilde{\theta}^{\dot{\beta}} = \tilde{\epsilon}^{\dot{\alpha}\dot{\beta}} \quad ; \quad (\text{A.43})$$

$$\partial_\alpha \theta^2 = 2\theta_\alpha \quad , \quad \tilde{\partial}_{\dot{\alpha}} \tilde{\theta}^2 = 2\tilde{\theta}_{\dot{\alpha}} \quad ; \quad (\text{A.44})$$

$$\partial^\alpha \theta^2 = 2\theta^\alpha \quad , \quad \tilde{\partial}^{\dot{\alpha}} \tilde{\theta}^2 = 2\tilde{\theta}^{\dot{\alpha}} \quad ; \quad (\text{A.45})$$

$$\partial^2 \theta^2 = -4 \quad , \quad \tilde{\partial}^2 \tilde{\theta}^2 = -4 \quad . \quad (\text{A.46})$$

Throughout this work, the bosonic derivatives are defined by

$$\not{\partial} \equiv \epsilon \sigma^\mu \partial_\mu \quad , \quad (\text{A.47})$$

$$\tilde{\not{\partial}} \equiv \tilde{\epsilon} \tilde{\sigma}^\mu \partial_\mu \quad . \quad (\text{A.48})$$

The superspace measures for the Atiyah-Ward space-time are

$$ds \equiv d^4 x d^2 \theta \quad , \quad d\tilde{s} \equiv d^4 x d^2 \tilde{\theta} \quad \text{and} \quad dv \equiv d^4 x d^2 \theta d^2 \tilde{\theta} \quad , \quad (\text{A.49})$$

where the following normalization conditions are taken :

$$\int d^2 \theta \theta^2 = 1 \quad \text{and} \quad \int d^2 \tilde{\theta} \tilde{\theta}^2 = 1 \quad . \quad (\text{A.50})$$

For any superfield,  $\Phi(x, \theta, \tilde{\theta})$ , it can be directly shown that

$$\int d^2 \theta \Phi = -\frac{1}{4} \partial^2 \Phi = -\frac{1}{4} D^2 \Phi \big|_{\theta=\tilde{\theta}=0} \quad , \quad \int d^2 \tilde{\theta} \Phi = -\frac{1}{4} \tilde{\partial}^2 \Phi = -\frac{1}{4} \tilde{D}^2 \Phi \big|_{\theta=\tilde{\theta}=0} \quad (\text{A.51})$$

$$\text{and} \quad \int d^2 \theta d^2 \tilde{\theta} \Phi = \frac{1}{16} \partial^2 \tilde{\partial}^2 \Phi = \frac{1}{16} D^2 \tilde{D}^2 \Phi \big|_{\theta=\tilde{\theta}=0} \quad . \quad (\text{A.52})$$

## B General conventions for $D=1+2$ and some rules for dimensional reduction

In Section 4, we have adopted the metric  $\eta_{mn}=\text{diag}(+, -, -)$ ,  $m, n=(0,1,2)$ , for  $D=1+2$ . The Dirac  $2\times 2$   $\gamma$ -matrices that satisfy the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{1}_2 \quad , \quad (\text{B.1})$$

are as follows

$$\gamma^m = i\sigma^m = -i\tilde{\sigma}^m = (\sigma_x, i\sigma_y, -i\sigma_z) \quad , \quad (\text{B.2})$$

where  $\sigma^m$  and  $\tilde{\sigma}^m$  are defined by eqs.(A.10) and (A.11). In  $D=1+2$ , the  $\gamma$ -matrices have an addition relation :

$$\gamma^m \gamma^n = \eta^{mn}\mathbb{1}_2 + i\epsilon^{mnl}\gamma_l \quad . \quad (\text{B.3})$$

We have the following generators of the  $\overline{SO(1,2)}$ -group in the spinor representation :

$$\Sigma^{kl} = \frac{1}{4}[\gamma^k, \gamma^l] \quad . \quad (\text{B.4})$$

This yields an important relation used in the computation of the supergauge sector action (49) :

$$\text{Tr}(\Sigma^{kl}\Sigma^{mn}) = \frac{1}{2}(\eta^{km}\eta^{ln} - \eta^{kn}\eta^{lm}) \quad . \quad (\text{B.5})$$

The charge conjugation matrix is found as

$$C = -i\epsilon = i\tilde{\epsilon} = \sigma_y \quad , \quad (\text{B.6})$$

where  $\epsilon$  and  $\tilde{\epsilon}$  are defined by eq.(A.7).

The charge-conjugated and the adjoint spinors are defined by

$$\psi^c = -C\bar{\psi}^T \quad \text{and} \quad \bar{\psi} \equiv \psi^\dagger \gamma^0 \quad . \quad (\text{B.7})$$

Some useful relations involving spinorial bilinears are listed below :

$$\begin{aligned} (\bar{\psi}\chi)^T &= \bar{\chi}^c \psi^c \quad , \\ (i\bar{\psi}\gamma^m\chi)^T &= -i\bar{\chi}^c \gamma^m \psi^c \quad , \\ (i\bar{\psi}\gamma^m\partial_m\chi)^T &= i\bar{\chi}^c \gamma^m \partial_m \psi^c \quad ; \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} (\bar{\psi}\chi)^* &= \bar{\psi}^c \chi^c \quad , \\ (i\bar{\psi}\gamma^m\chi)^* &= i\bar{\psi}^c \gamma^m \chi^c \quad , \\ (i\bar{\psi}\gamma^m\partial_m\chi)^* &= i\bar{\psi}^c \gamma^m \partial_m \chi^c \quad ; \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} (\bar{\psi}\chi)^\dagger &= \bar{\chi}\psi \quad , \\ (i\bar{\psi}\gamma^m\chi)^\dagger &= -i\bar{\chi}\gamma^m\psi \quad , \\ (i\bar{\psi}\gamma^m\partial_m\chi)^\dagger &= i\bar{\chi}\gamma^m\partial_m\psi \quad . \end{aligned} \quad (\text{B.10})$$

As  $\theta$  is a Majorana spinor ( $\theta^c = \theta$ ), by using the eqs.(B.6) and (B.7), we found that

$$\theta_a = (\bar{\theta}C)_a \quad , \quad \bar{\theta}_a = -(C\theta)_a \quad ; \quad (\text{B.11})$$

$$\theta_a \theta_b = -\frac{1}{2} \bar{\theta} \theta C_{ab} \quad , \quad \bar{\theta}_a \bar{\theta}_b = \frac{1}{2} \bar{\theta} \theta C_{ab} \quad ; \quad (\text{B.12})$$

$$\bar{\theta}_a \theta_b = \frac{1}{2} \bar{\theta} \theta \delta_{ab} \quad , \quad (\text{B.13})$$

$$\bar{\theta} \theta \equiv \bar{\theta}_a \theta_a \quad . \quad (\text{B.14})$$

The fermionic derivatives in  $D=1+2$  are defined as

$$\partial_a \theta_b = \delta_{ab} \quad , \quad \bar{\partial}_a \bar{\theta}_b = \delta_{ab} \quad ; \quad (\text{B.15})$$

$$\partial_a \bar{\theta}_b = C_{ab} \quad , \quad \bar{\partial}_a \theta_b = C_{ab} \quad ; \quad (\text{B.16})$$

$$\partial_a \bar{\theta} \theta = -2 \bar{\theta}_a \quad , \quad \bar{\partial}_a \bar{\theta} \theta = 2 \theta_a \quad ; \quad (\text{B.17})$$

$$(\bar{\partial} \partial)(\bar{\theta} \theta) = -4 \quad . \quad (\text{B.18})$$

The superspace measure for  $D=1+2$ <sup>8</sup> space-time dimensions is

$$d\hat{v} \equiv d^3 \hat{x} d^2 \theta \quad ; \quad (\text{B.19})$$

it respects the normalization condition

$$\int d^2 \theta \bar{\theta} \theta = 1 \quad . \quad (\text{B.20})$$

The covariant superderivatives satisfy the following algebra

$$\{D_a, D_b\} = 2i(\gamma^m C)_{ab} \partial_m \quad , \quad (\text{B.21})$$

$$\{\bar{D}_a, \bar{D}_b\} = -2i(C\gamma^m)_{ab} \partial_m \quad , \quad (\text{B.22})$$

$$\{D_a, \bar{D}_b\} = 2i\gamma_{ab}^m \partial_m \quad , \quad (\text{B.23})$$

where they are given by

$$D_a = \bar{\partial}_a - i(\gamma^m \theta)_a \partial_m \quad , \quad \bar{D}_a = -\partial_a + i(\bar{\theta} \gamma^m)_a \partial_m \quad . \quad (\text{B.24})$$

For any superfield,  $\Phi(\hat{x}, \theta)$ , it can be directly shown that

$$\int d^2 \theta \Phi = -\frac{1}{4} \bar{\partial} \partial \Phi = -\frac{1}{4} \bar{D} D \Phi |_{\theta=0} \quad . \quad (\text{B.25})$$

The relation between the  $\gamma$ -matrices in  $D=2+2$  and in  $D=1+2$  are listed below,

$$\epsilon \sigma^\mu = (C \gamma^m, iC) \quad (\text{B.26})$$

$$\tilde{\epsilon} \tilde{\sigma}^\mu = (C \gamma^m, -iC) \quad , \quad (\text{B.27})$$

---

<sup>8</sup>The hat symbol ( $\hat{\phantom{x}}$ ) over the 3-dimensional space-time coordinates is to distinguish from the Atiyah-Ward ones.

where the left hand side is written in terms of the  $D=2+2$  quantities, whereas the right side in terms of  $D=1+2$  ones.

To perform the dimensional reduction *à la* Scherk of the massive  $N=1$  super-QED $_{2+2}$ -action (30) to  $D=1+2$ , use has been made of the rules presented below. One uses the trivial dimensional reduction where the time-derivative,  $\partial_3$ , of all component fields vanishes,  $\partial_3 \mathcal{F}=0$ . Also, it was assumed that  $B_\mu$  is reduced in the following manner:  $B^\mu=(B^m, \phi)$ , where  $\phi$  is a complex scalar field. Note that the Weyl spinors in  $D=2+2$  transform into Dirac ones after the dimensional reduction is carried out to  $D=1+2$ . Now, we list the the following rules for the dimensional reduction (DR) carried out ; they read :

$$G_{\mu\nu}^* G^{\mu\nu} \xrightarrow{\text{DR}} G_{mn}^* G^{mn} + 2\partial_m \phi^* \partial^m \phi \quad , \quad (\text{B.28})$$

$$\psi^c \tilde{\phi} \tilde{\chi} \rightarrow \bar{\psi} \gamma^m \partial_m \chi \quad , \quad (\text{B.29})$$

$$\tilde{\chi}^c \tilde{\phi} \psi \rightarrow \bar{\chi} \gamma^m \partial_m \psi \quad , \quad (\text{B.30})$$

$$iB_\mu \psi^c \sigma^\mu \tilde{\chi} \rightarrow iB_m \bar{\psi} \gamma^m \chi - \phi \bar{\psi} \chi \quad , \quad (\text{B.31})$$

$$B_\mu B^\mu A^* \rightarrow B_m B^m A^* \quad , \quad (\text{B.32})$$

$$iA^* \tilde{\chi} \tilde{\rho} \rightarrow A^* \bar{\chi}^c \rho \quad , \quad (\text{B.33})$$

$$iB \psi^c \lambda \rightarrow -B \bar{\psi} \lambda \quad , \quad (\text{B.34})$$

$$B_\mu B^\mu A^* B \rightarrow B_m B^m A^* B + \phi^2 A^* B \quad , \quad (\text{B.35})$$

$$i\psi_+ \psi_- \rightarrow -\bar{\psi}_+^c \psi_- \quad , \quad (\text{B.36})$$

$$i\tilde{\chi}_+ \tilde{\chi}_- \rightarrow \bar{\chi}_+^c \chi_- \quad , \quad (\text{B.37})$$

where the fields in the left hand side are fields living in  $D=2+2$  and in the other side appear fields living in  $D=1+2$ .

## Acknowledgements

The authors are deeply indebted to Dr. J.A. Helayël-Neto for suggesting the problem, for all the exhaustive discussions and the careful reading of the manuscript. Dr. O. Piguet, Dr. S.P. Sorella and Dr. L.P. Colatto are acknowledged for helpful discussions; Mrs. M.N.P. Magalhães is acknowledged for participation at an early stage of this work. Thanks are also due to our colleagues at DCP, in special to Dr. S.A. Dias, for encouragement. CNPq-Brazil is acknowledged for invaluable financial help. Finally, the authors acknowledge the Organizing Committee of the *Spring School and Workshop on String Theory, Gauge Theory and Quantum Gravity '95* held at the *International Centre for Theoretical Physics (ICTP)*.

## References

- [1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, *Phys. Lett.* **B59** (1975) 85 ;  
R. S. Ward, *Phys. Lett.* **B61** (1977) 81 ;  
M. F. Atiyah and R. S. Ward, *Comm. Math. Phys.* **55** (1977) 117 ;  
E. F. Corrigan, D. B. Fairlie, R. C. Yates and P. Goddard, *Comm. Math. Phys.* **58** (1978)

- 223 ;  
E. Witten, *Phys. Rev. Lett.* **38** (1977) 121.
- [2] M. F. Atiyah, *unpublished* ;  
R. S. Ward, *Phil. Trans. R. London* **A315** (1985) 451 ;  
N. J. Hitchin, *Proc. London Math. Soc.* **55** (1987) 59.
- [3] H. Ooguri and C. Vafa, *Mod. Phys. Lett.* **A5** (1990) 1389 ;  
H. Ooguri and C. Vafa, *Nucl. Phys.* **B361** (1991) 469 ;  
H. Ooguri and C. Vafa, *Nucl. Phys.* **B367** (1991) 83.
- [4] S. J. Gates Jr. and H. Nishino, *Mod. Phys. Lett.* **A7** (1992) 2543.
- [5] S. J. Gates Jr., S. V. Ketov and H. Nishino, *Phys. Lett.* **B307** (1993) 323.
- [6] S. J. Gates Jr., S. V. Ketov and H. Nishino, *Phys. Lett.* **B297** (1992) 99 ;  
S. J. Gates Jr., S. V. Ketov and H. Nishino, *Nucl. Phys.* **B393** (1993) 149.
- [7] H. Nishino, *Mod. Phys. Lett.* **A9** (1994) 3255.
- [8] W. Siegel, *Nucl. Phys.* **B156** (1979) 135 ;  
J. Schonfeld, *Nucl. Phys.* **B185** (1981) 157 ;  
R. Jackiw and S. Templeton, *Phys. Rev.* **D23** (1981) 2291 ;  
S. Deser and R. Jackiw, *Phys. Lett.* **B139** (1984) 371 ;  
S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975 ;  
S. Deser, R. Jackiw and S. Templeton, *Ann. Phys. (N.Y.)* **140** (1982) 372.
- [9] S. Weinberg, *Understanding the Fundamental Constituents of Matter*, ed. A. Zichichi, Plenum Press (New York, 1978) ;  
A. Linde, *Rep. Progr. Phys.* **42** (1979) 389 ;  
D. Gross, R. D. Pisarski and L. Yaffe *Rev. Mod. Phys.* **53** (1981) 43.
- [10] O. Foda, *Nucl. Phys.* **B300** (1988) 611 ;  
Y. H. Chen, F. Wilczek, E. Witten and B. I. Halperin, *Int. J. Mod. Phys.* **B3** (1989) 1001 ;  
J. D. Lykken, *Chern-Simons and Anyonic Superconductivity*, talk given at the fourth annual Superstring Workshop, “Strings 90”, Texas A & M University (Texas, March 1990).
- [11] N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B266** (1991) 163 ;  
N. Dorey and N. E. Mavromatos, *Nucl. Phys.* **B386** (1992) 614.
- [12] R. D. Pisarski, *Phys. Rev.* **D29** (1984) 2423 ;  
T. W. Appelquist, M. Bowick, D. Karabali and L. C. R. Wijewardhana, *Phys. Rev.* **D33** (1986) 3704 ;  
R. Mackenzie and F. Wilczek, *Int. J. Mod. Phys.* **A3** (1988) 2827 ;  
G. W. Semenoff and P. Sodano, *Nucl. Phys.* **B328** (1989) 753 ;  
G. W. Semenoff and L. C. R. Wijewardhana, *Phys. Rev. Lett.* **63** (1989) 2633 ;  
A. Kovner and B. Rosenstein, *Mod. Phys. Lett.* **A5** (1990) 2661.
- [13] F. Delduc, C. Lucchesi, O. Piguet and S. P. Sorella, *Nucl. Phys.* **B346** (1990) 313 ;  
A. Blasi, O. Piguet and S. P. Sorella, *Nucl. Phys.* **B356** (1991) 154 ;  
C. Lucchesi and O. Piguet, *Nucl. Phys.* **B381** (1992) 281.

- [14] J. Scherk, *Extended Supersymmetry and extended Supergravity Theories*, Recent Developments in Gravitation, Cargèse 1978, Ed. M. Lévy and S. Deser, Plenum Press ;  
L. Brink, J. H. Schwarz and J. Scherk, *Nucl. Phys.* **B121** (1977) 77 ;  
F. Gliozzi, J. Scherk and D. Olive, *Nucl. Phys.* **B122** (1977) 253 ;  
J. Scherk and J. H. Schwarz, *Nucl. Phys.* **B153** (1979) 61.
- [15] M. F. Sohnius, *Phys. Rep.* **128** (1985) 39 ;  
P. van Nieuwenhuizen, *An Introduction to Simple Supergravity and the Kaluza-Klein Program*, Relativity, Groups and Topology II, Les Houches 1984, Ed. B.S. de Witt and R. Stora, North-Holland.
- [16] M. A. De Andrade and O. M. Del Cima, *Phys. Lett.* **B347** (1995) 95.
- [17] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace*, Benjamin/Cummings (Reading, 1983) ;  
J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Univ. Press (Princeton, 1983).
- [18] O. Piguet and K. Sibold, *Renormalized Supersymmetry*, Birkhäuser Press (Boston, 1986).
- [19] A. Salam and J. S. Strathdee, *Nucl. Phys.* **B76** (1974) 477 ;  
A. Salam and J. S. Strathdee, *Nucl. Phys.* **B80** (1974) 499 ;  
A. Salam and J. S. Strathdee, *Phys. Rev.* **D11** (1975) 1521.
- [20] A. Salam and J. S. Strathdee, *Phys. Lett.* **B51** (1974) 353.
- [21] J. Wess and B. Zumino, *Nucl. Phys.* **B70** (1974) 39 ;  
J. Wess and B. Zumino, *Phys. Lett.* **B49** (1974) 52 ;  
J. Wess and B. Zumino, *Nucl. Phys.* **B78** (1974) 1 ;  
S. Ferrara and B. Zumino, *Nucl. Phys.* **B79** (1974) 413.
- [22] O. Piguet, J. A. Helayël-Neto and S. P. Sorella, *private communications*.
- [23] J. A. Helayël-Neto, *private communications*.
- [24] A. Parkes, *Phys. Lett.* **B286** (1992) 265 ;  
W. Siegel, *Phys. Rev.* **D47** (1993) 2504.
- [25] M. A. De Andrade, O. M. Del Cima and L. P. Colatto, *N=1 super-Chern-Simons coupled to parity-preserving matter from Atiyah-Ward space-time*, hep-th 9506146, *submitted for publication*.
- [26] M. A. De Andrade, *Mod. Phys. Lett.* **A10** (1995) 961.
- [27] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley-Interscience (1974).